Some Generalized Multivariate G-Function Distributions of Matrix Argument and Statistical Properties and Integral Transform

Dr A. K. Vyas
Department of Mathematics
JodhpurNationalUniversity
Jodhpur

Abstract:- In the present paper we have discussed various generalized multivariate G-Function distributions of matrix argument and statistical properties and integral transform few known or new properties such as characteristic function, moment generating function, moment, mean and variance and Mellin and Laplace transforms of the distribution function and some new table are also established for distribution function.

Introduction

A unified approach to generate probability distributions with the help of special functions technique was initiated by Mathai and Saxena(1966), Saxena and Mishra (1991) have investigated a more comprehensive from of univariate distributions by taking a generalized Laplace integral incorporating ${}_2F_2$ function. From this result wide classes of distributions have been obtained as particular cases. Sethi and Poonam (1992) have investigated a more comprehensive form of univariate distribution by taking a generalized Laplace integral involving ${}_2F_2$ of matrix argument. From this result, wide classes of distributions have been obtained as particular cases.

In this paper, a generalized multivariate G-function distribution are introduced from which almost all well-known multivariate and univariate G-function distributions can be obtained as special cases. It depends on the multivariate Laplace integral, our distribution in analogues to the generalized probability distribution given by Poonam and Sethi P.L. (1992C) and has applications in multivariate analysis. It may provide some interesting special random variables obtained in various fields of statistical studies particularly in the field of biosciences, agricultural, social and behavioral sciences along with the resent techniques utilized in the geo-statistical and atmospheric sciences.

In this paper expression for the characteristic function. The Moment Generating Function and the r^{th} Moment about the origin, mean, variance, Mellin and Laplace Transform have been worked out. In what follows, the argument and the parameter and restricted to take only those values for which the density functions are non-negative and have a meaning.

THE DISTRIBUTION

In the multivariate Laplace type integral

$$I = \int_{X>0} etr(-BX)|X|^{\alpha-\frac{m+1}{2}}\phi(X)dX$$

Taking $\phi(X) = G^{p,q}_{r,s}\left[RX\middle|\begin{matrix}a_1\ldots\ldots\ldots a_r\\ b_1\ldots\ldots b_s\end{matrix}\right]$

The integral reduces to

$$I = |B|^{-\alpha} G^{p,q+1}_{r+1,s}\left[RB^{-1} \middle| \begin{matrix} \frac{m+1}{2} - \alpha, & a_{1....} a_r \\ b_{1..........................} b_s & \end{matrix} \right] .... (2.1)$$

For $\mathrm{Re}\left(-\alpha + \min b_j\right) > \frac{m-1}{2} (j = 1,\ldots,p)$ and B is positive definite symmetric matrix and R is an arbitrary complex symmetric $m \times m$ matrix.S

The result (2.1) is a direct consequence of the result [Mathai and Saxena 1978,P.114]

Thus the function

$$f(X) = f\left(X; \alpha_1, a_1 \ldots, a_r, b_1, .b_s; B, R\right)$$

$$= \frac{etr\left(-BX\right)|X|^{\alpha - \frac{(m+1)}{2}} G^{p,q}_{r,s}\left[RX \middle| \begin{matrix} a_1 \ldots a_r \\ b_{1......} b_s \end{matrix} \right]}{|B|^{-\alpha} G^{p,q+1}_{r+1,s}\left[RB^{-1} \middle| \begin{matrix} \frac{m+1}{2} - \alpha, a_{1,.....} a_r \\ b_{1,.............................} b_s \end{matrix} \right]}$$

Where $\mathrm{Re}\left(-\alpha + \min b_j\right) > \frac{m-1}{2}$

= 0, elsewhere ………………………… (2.2)

Provides a probability density function(p.d.f.)

Special cases

(i) Replacing R=I letting B tends to null matrix and using the result due to Mathai(1976)

$$\int_{X>0} |X|^{\alpha - \frac{(m+1)}{2}} G^{p,q}_{r,s}\left[X \middle| \begin{matrix} a_1 \ldots a_r \\ b_{1,....} b_s \end{matrix} \right] dX$$

$$= \frac{\prod_{j=1}^{p} \Gamma m(b_j + \rho) \prod_{j=1}^{q} \Gamma m\left(\frac{m+1}{2} - a_j - \alpha\right)}{\prod_{j=p+1}^{s} \Gamma m\left(\frac{m+1}{2} - b_j - \alpha\right) \prod_{j=q+1}^{r} \Gamma m\left(a_j + \alpha\right)}$$

Where

$$\mathrm{Re}(b_j + \alpha) > \frac{m-1}{2}; (j = 1,..,m)$$
$$\mathrm{Re}(a_j + \alpha) < \frac{m+1}{2}; (j = 1,..n)$$
…………..(2.3)

In (2.2), we get

$$f(X)=\left[\frac{\prod_{j=1}^{p}\Gamma\mathrm{m}(\mathrm{b_j}+\rho)\prod_{j=1}^{q}\Gamma m\left(\frac{m+1}{2}-a_j-\alpha\right)}{\prod_{j=p+1}^{s}\Gamma m\left(\frac{m+1}{2}-b_j-\alpha\right)\prod_{j=q+1}^{r}\Gamma m\left(a_j+\alpha\right)}\right]^{-1}|X|^{\alpha-\frac{(m+1)}{2}}G_{r,s}^{p,q}[*]$$

Where $G_{r,s}^{p,q}[*]=G_{r,s}^{p,q}\left[X\middle|\begin{matrix}a_{1,}\dots,a_r\\ b_1,\dots,b_s\end{matrix}\right]$

Where

$$\mathrm{Re}(b_j+\alpha)>\frac{m-1}{2};(j=1,..,m)$$

$$\mathrm{Re}(a_j+\alpha)<\frac{m+1}{2};(j=1,...n),\ X=X'>0 \ \ldots\ldots\ldots\ldots..(2.4)$$

$$=0, elsewhere$$

(ii) Putting $p=1, q=0, r=0, s=1, B=I,$ then (2.1) takes the form

$$I=\int_{X>0} etr(-X)|X|^{\alpha-\frac{(m+1)}{2}}G_{0,1}^{1,0}\left[RX|a\right]dX \ \ldots\ldots\ldots.(2.5)$$

We know that $G_{0,1}^{1,0}\left[RX|a\right]=|R|^{a}|X|^{a}e^{-trRX}$

Where $X=X'>0$ ………………….(2.6)

Use of (2.5) and (2.6) we have

$$I=|R|^{a}\int_{X>0}e^{-tr(R+I)X}|X|^{\alpha+a-\frac{(m+1)}{2}}dX$$

The integral reduces to

$$=|R|^{a}\Gamma m(\alpha+a)|I+R|^{-(\alpha+a)} \ \ldots\ldots\ldots\ldots\ldots.(2.6)$$

(2.2) takes the form

$$f(X)=\frac{e^{-tr(I+R)X}|X|^{(\alpha+a)-\frac{(m+1)}{2}}}{\Gamma m(\alpha+a)|I+R|^{-(\alpha+a)}}$$

$$\mathrm{Re}(\alpha+a)>\frac{m-1}{2}, \mathrm{Re}(I+R)>0, X=X'>0$$

Where

$$=0, elsewhere\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.7)$$

Which is a gama distribution.

Taking $(\alpha+a)=\frac{m+1}{2}$, (2.7) take the form

$$f(X)=\frac{e^{-tr(I+R)X}}{\Gamma m\left(\frac{m+1}{2}\right)|I+R|^{-\frac{(m+1)}{2}}}$$

Where

$$\operatorname{Re}(I+R)>0,\ X=X'>0$$

=0 elsewhere ……………..(2.8)

Taking $(\alpha+a)=\frac{n}{2},(I+R)=\frac{1}{2}T^{-1},$ (2.7) yields Wishart distribution with scale matrix T and n of freedom

$$f(X)=\frac{e^{-tr\frac{(T^{-1}X)}{2}}|X|^{\frac{n}{2}-\frac{(m+1)}{2}}}{\Gamma m\left(\frac{n}{2}\right)\left|\frac{1}{2}T^{-1}\right|^{-\frac{n}{2}}}$$

$$=\frac{2^{\frac{-n}{2}}|X|^{\frac{(n-m-1)}{2}}etr\left(\frac{-1}{2}T^{-1}X\right)}{\Gamma m\left(\frac{n}{2}\right)|T|^{\frac{n}{2}}}$$

For $X=X'>0,T>0,m\leq n$

= $0, elsewhere$ ……………………………………….(2.9)

THE CHARACTERISTIC FUNCTION

The characteristic function $\Phi_X(T)$ of X is given by

$$\Phi_X(T)=E[\exp\{itr(XT)\}]$$

$$=\int_{X>0}\exp[itr(XT)]f(X)dX\ldots\ldots\ldots\ldots\ldots\ldots(2.10)$$

Where $i=(-1)^{1/2},T$ is $m\times m$ symmetric matrix of characteristic function variables and E denotes mathematical expectation.

By Virtue of (2.10), we get characteristic function as

$$\Phi_X(T)=\frac{|B|-iT|^{-\alpha}G_{r+1,s}^{p,q+1}\left[R(B-iT)^{-1}\left|\begin{matrix}\frac{m+1}{2}-\alpha,a_1,\ldots,a_r\\ b_1,\ldots\ldots,b_s\end{matrix}\right.\right]}{|B|^{-\alpha}G_{r+1,s}^{p,q+1}\left[RB^{-1}\left|\begin{matrix}\frac{m+1}{2}-\alpha,a_1,\ldots,a_r\\ b_1,\ldots\ldots,b_s\end{matrix}\right.\right]}$$

Where $\operatorname{Re}(\alpha)>\frac{m+1}{2},\operatorname{Re}(\alpha+\min b_j)>\frac{m-1}{2},\operatorname{Re}(B)>0,\operatorname{Re}(B)>\operatorname{Re}(R)$ …(2.11)

THE MOMENT GENERATING FUNCTION (m.g.h.)

$$E[etr(XT)] = \int_{X>0} etr(XT) f(X) dX \quad \ldots\ldots\ldots\ldots\ldots(2.12)$$

By virtue of (2.2),we get moment generating function as

$$E[etr(XT)] = \frac{|B-T|^{-\alpha} G^{p,q+1}_{r+1,s}\left[R(B-T)^{-1}\left|\begin{matrix}\frac{m+1}{2}-\alpha, a_1,\ldots,a_r\\ b_1,\ldots\ldots,b_s\end{matrix}\right.\right]}{|B|^{-\alpha} G^{p,q+1}_{r+1,s}\left[RB^{-1}\left|\begin{matrix}\frac{m+1}{2}-\alpha, a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\right.\right]}$$

Where $\mathrm{Re}(\alpha) > \frac{m-1}{2}, \mathrm{Re}(\alpha + \min b_j) > \frac{m-1}{2}, \mathrm{Re}(B) > 0, \mathrm{Re}(B) > \mathrm{Re}(R)$ …(2.13)

THE MOMENT OF THE DISTRIBUTION

The r^{th} order moment about origin of the matrix random variable X with p.d.f. (2.2) is given by

$$E\left[|X|^r\right] = \int_{X>0} |X|^r f(X) dX \quad \ldots\ldots\ldots\ldots\ldots(2.14)$$

$$= \frac{|B|^{-r} G^{p,q+1}_{r+1,s}\left[RB^{-1}\left|\begin{matrix}\frac{m+1}{2}-\alpha-r, a_1,\ldots,a_r\\ b_1,\ldots\ldots,b_s\end{matrix}\right.\right]}{G^{p,q+1}_{r+1,s}\left[RB^{-1}\left|\begin{matrix}\frac{m+1}{2}-\alpha, a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\right.\right]}$$

Where the various parameter are governed by same restrictions as mentioned in (2.15)

MEAN AND VARIANCE OF THE DISTRIBUTION BY DEFINITIONS

$$E[|X|] = \int_{X>0} |X| f(X) dX$$

$$= \frac{|B|^{-1} G^{p,q+1}_{r+1,s}\left[RB^{-1}\left|\begin{matrix}\frac{m+1}{2}-\alpha-1, a_1,\ldots,a_r\\ b_1,\ldots\ldots,b_s\end{matrix}\right.\right]}{G^{p,q+1}_{r+1,s}\left[RB^{-1}\left|\begin{matrix}\frac{m+1}{2}-\alpha, a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\right.\right]} \quad \ldots\ldots(2.16)$$

By definition , variance = $E(|X|^2) - [E(|X|)]^2$

$$= \frac{|B|^{-2} G_{r+1,s}^{p,q+1}\left[RB^{-1} \left| \begin{matrix} \frac{m+1}{2} - \alpha - 2, a_1, \ldots, a_r \\ b_1, \ldots\ldots, b_s \end{matrix} \right. \right]}{G_{r+1,s}^{p,q+1}\left[RB^{-1} \left| \begin{matrix} \frac{m+1}{2} - \alpha, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right. \right]} \quad \ldots\ldots(2.17)$$

$$\text{Variance } = \frac{|B|^{-2}\left[G_{r+1,s}^{p,q+1}[-]\, G_{r+1,s}^{p,q+1}[=] - \left\{ G_{r+,s}^{p,q+1}[\equiv] \right\}^2 \right]}{\left[G_{r+1,s}^{p,q+1}[=] \right]^2}$$

Where $G_{r+1,s}^{p,q+1}[-] = G_{r+1,s}^{p,q+1}\left[RB^{-1} \left| \begin{matrix} \frac{m+1}{2} - \alpha - 2, a_1, \ldots, a_r \\ b_1, \ldots\ldots, b_s \end{matrix} \right. \right]$

$$G_{r+1,s}^{p,q+1}[=] = G_{r+1,s}^{p,q+1}\left[RB^{-1} \left| \begin{matrix} \frac{m+1}{2} - \alpha, a_1, \ldots, a_r \\ b_1, \ldots\ldots, b_s \end{matrix} \right. \right]$$

$$G_{r+1,s}^{p,q+1}[\equiv] = G_{r+1,s}^{p,q+1}\left[RB^{-1} \left| \begin{matrix} \frac{m+1}{2} - \alpha - 1, a_1, \ldots, a_r \\ b_1, \ldots\ldots, b_s \end{matrix} \right. \right] \quad \ldots\ldots(2.18)$$

Where the various parameter are governed by same restriction as mentioned in (2.2)

MELLIN AND LAPLACE TRANSFORM OF THE DISTRIBUTION

By definition of Mellin transform is

$$M\left[f(X)\right] = \int_{X>0} |X|^{\delta - \frac{(m+1)}{2}} f(X)\,dX$$

$$= \frac{|B|^{-\delta + \frac{(m+1)}{2}} G_{r+1,s}^{p,q+1}\left[RB^{-1} \left| \begin{matrix} \frac{m+1}{2} - \left(\alpha + \delta - \frac{m+1}{2} \right), a_1, \ldots, a_r \\ b_1, \ldots\ldots, b_s \end{matrix} \right. \right]}{G_{r+1,s}^{p,q+1}\left[RB^{-1} \left| \begin{matrix} \frac{m+1}{2} - \alpha, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right. \right]} \quad \ldots\ldots(2.19)$$

By definition of Laplace transform is

$$L\left[f(X)\right] = \int_{X>0} etr(-TX) f(X)\,dX$$

$$L[f(X)] = \frac{|B+T|^{-\alpha} G^{p,q+1}_{r+1,s}\left[R(B+T)^{-1} \middle| \begin{matrix} \frac{m+1}{2} - \alpha, a_1, \ldots, a_r \\ b_1, \ldots\ldots, b_s \end{matrix} \right]}{|B|^{-\alpha} G^{p,q+1}_{r+1,s}\left[RB^{-1} \middle| \begin{matrix} \frac{m+1}{2} - \alpha, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right]}$$

Where $\mathrm{Re}(\alpha) > \frac{m-1}{2}, \mathrm{Re}(\alpha + \min b_j) > \frac{m-1}{2}$

$\mathrm{Re}(B) > 0, \mathrm{Re}(B) > \mathrm{Re}(R)\ \mathrm{Re}(B+T) > 0$

=0, elsewhere …………(2.20)

Special function of statistical distributions.

$\Phi_X(T)\ E[etr|XT], E[|X|^r]$, Variance $M[f(X)]$, $L[f(X)]$ are defined already through equation (2.10),(2.12),(2.14),(2.17),(2.19),(2.20) respectively.

TABLE 1

$$f(X) = \frac{e^{-tr(1+R)X} |X|^{(\alpha+a)-\frac{(m+1)}{2}}}{\Gamma m(\alpha+a)|1+R|^{-(\alpha+a)}}$$

Where $\mathrm{Re}(\alpha + a) > \frac{m-1}{2}, \mathrm{Re}(1+R) > 0, X = X' > 0$

$\Phi_X(T)$	$E[etr(XT)]$	$E[\lvert X\rvert^r]$
$\frac{\lvert I+(R-T)\rvert^{-(\alpha+a)}}{\lvert I+R\rvert^{-(\alpha+a)}}$	$\frac{\lvert I+(R-T)\rvert^{-(\alpha+a)}}{\lvert I+R\rvert^{-(\alpha+a)}}$	$\frac{\Gamma m(\alpha+a+r)\lvert I+R\rvert^{-r}}{\Gamma m(\alpha+a)}$
Variance= $E(\lvert X\rvert^2) - [E(\lvert X\rvert)]^2$	$M[f(X)]$	$L[f(X)]$
$\frac{\Gamma m(\alpha+a+2)\lvert I+R\rvert^{-2}}{\Gamma m(\alpha+a)} - \left[\frac{\Gamma m(\alpha+a+1)\lvert I+R\rvert^{-1}}{\Gamma m(\alpha+a)}\right]^2$	$\frac{\Gamma m\left(\alpha+a+s-\frac{m+1}{2}\right)\lvert I+R\rvert^{-s+\frac{m+1}{2}}}{\Gamma m(\alpha+a)}$	$\frac{\lvert I+(R+T)\rvert^{-(\alpha+a)}}{\lvert I+R\rvert^{-(\alpha+a)}}$

TABLE 2

$$f(X)=\frac{e^{-tr(1+R)X}}{\Gamma m\left(\frac{m+1}{2}\right)\left|I+R\right|^{-\frac{(m+1)}{2}}}$$

$$Where:-\mathrm{Re}(I+R)>0,\ X=X'>0$$

$\Phi_X(T)$	$E[etr(XT)]$	$E[\lvert X\rvert^r]$
$\frac{\lvert I+(R-T)\rvert^{-\frac{(m+1)}{2}}}{\lvert I+R\rvert^{-\frac{(m+1)}{2}}}$	$\frac{\lvert I+(R-T)\rvert^{-\frac{(m+1)}{2}}}{\lvert I+R\rvert^{-\frac{(m+1)}{2}}}$	$\frac{\Gamma m\left(\frac{m+1}{2}+r\right)\lvert I+R\rvert^{-r}}{\Gamma m\left(\frac{m+1}{2}\right)}$
Variance= $E(\lvert X\rvert^2)-[E(\lvert X\rvert)]^2$	$M[f(X)]$	$L[f(X)]$
$\frac{\Gamma m\left(\frac{m+1}{2}+2\right)\lvert I+R\rvert^{-2}}{\Gamma m\left(\frac{m+1}{2}\right)}-\left[\frac{\Gamma m\left(\frac{m+1}{2}+1\right)\lvert I+R\rvert^{-1}}{\Gamma m\left(\frac{m+1}{2}\right)}\right]^2$	$\frac{\Gamma m(s)\lvert I+R\rvert^{-s+\frac{m+1}{2}}}{\Gamma m\left(\frac{m+1}{2}\right)}$	$\frac{\lvert I+(R+T)\rvert^{-\left(\frac{m+1}{2}\right)}}{\lvert I+R\rvert^{-\left(\frac{m+1}{2}\right)}}$

TABLE 3

$$f(X)=\frac{2^{-\frac{n}{2}}\left|X\right|^{\frac{(n-m-1)}{2}}etr\left(-\frac{1}{2}T^{-1}X\right)}{\Gamma m\left(\frac{n}{2}\right)\left|T\right|^{\frac{n}{2}}}$$

For $X=X'>0, T>0, m\le n$

$\Phi_X(T)$	$E[etr(XT)]$	$E[\lvert X\rvert^r]$
$\frac{\lvert s^{-1}-2iT\rvert^{-\frac{n}{2}}}{\lvert s\rvert^{-\frac{n}{2}}}$	$\frac{\lvert(s^{-1}-2T)\rvert^{-\frac{n}{2}}}{\lvert s\rvert^{-\frac{n}{2}}}$	$\frac{2^r\,\Gamma m\left(\frac{n}{2}+r\right)\lvert s\rvert^r}{\Gamma m\left(\frac{n}{2}\right)}$
Variance= $E(\lvert X\rvert^2)-[E(\lvert X\rvert)]^2$	$M[f(X)]$	$L[f(X)]$
$\frac{2^2\,\Gamma m\left(\frac{n}{2}+2\right)\lvert s\rvert^2}{\Gamma m\left(\frac{n}{2}\right)}-\left[\frac{2^1\,\Gamma m\left(\frac{n}{2}+1\right)\lvert s\rvert^1}{\Gamma m\left(\frac{n}{2}\right)}\right]^2$	$\frac{2^{s-\frac{m+1}{2}}\,\Gamma m\left(\frac{n}{2}+s-\frac{m+1}{2}\right)\lvert s\rvert^{s-\frac{m+1}{2}}}{\Gamma m\left(\frac{n}{2}\right)}$	$\frac{\lvert s^{-1}+2T\rvert^{-\frac{n}{2}}}{\lvert s\rvert^{\frac{n}{2}}}$

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