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Distribution Properties of Gauss Hypergeometric Function of Matrix Variable

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Abstract:- In this paper we establish distribution properties involving gauss hypergeometric function and finding moment generating function, r^{th} moment about origin, mean, variance, mellian, andlaplace transform have been worked out. The argument and parameter are restricted to take only those values for which the density functions are non negative and have meaning. All the matrices considered are real positive, definite and symmetric matrices of order $p \times p$.

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1. INTRODUCTION

LAPLACE AND INVERSE LAPLACE TRANSFORMS

Laplace and inverse Laplace transforms of matrices argument are respectively given by equations

$$(1.1) \qquad L_T[f(Z)] = \int_{T>0} e^{tr(-TZ)} f(T)dT = \Phi(Z)$$

.

And

$$(1.2) \qquad \frac{2^{p(p-1)/2}}{(2\pi i)^{p(p+1)/2}} \int_{\mathrm{Re}(Z)>0} e^{tr(TZ)} \Phi(Z)dZ = \begin{matrix} f(T), T>0 \\ 0, elsewhere \end{matrix}$$

Where $\Phi(Z)$ is complex analytic function and integral is taken over Z=X+i Y with fixed $X > X_0$ and Y over the space $S_p{}^*$.For the conditions and details the readers see to Mathai [2],[3] Mathai and saxena[1],Seemon Thomas, Alex Thannippara and Mathai[4]Sharma [6] obtain the solution of the integral equation of matrix argument and generalized stieltjes transform of matrixargument.

2. Main results

Theorem1:

$$I = \int_{X>O} etr(-BX)|X|^{\alpha-\frac{(p+1)}{2}} \phi(X)dX$$

If we take,

$$\phi(X) = \frac{\Gamma_P(\gamma+\nu+\frac{(p+1)}{2})}{\Gamma_P(\nu+\frac{(p+1)}{2})} {}_2F_1(-\gamma, \nu+\delta+\gamma+\frac{p+1}{2}; \nu+\frac{p+1}{2}; -RX)$$

$$= P_\gamma^{\nu,\delta}(X)$$

given integral reduce to

$$(2.1) \quad I = |B|^{-\alpha} \frac{\Gamma_P(\gamma+\nu+\frac{(p+1)}{2})\Gamma_P(\alpha)}{\Gamma_P(\nu+\frac{(p+1)}{2})} {}_3F_1(-\gamma, \nu+\delta+\gamma+\frac{p+1}{2}; \alpha; \nu+\frac{p+1}{2}; -RB^{-1})$$

thus the Function

$$(2.2) \quad f(X) = \frac{|X|^{\alpha-\frac{(p+1)}{2}} etr(-BX) {}_2F_1(-\gamma, \nu+\delta+\gamma+\frac{p+1}{2}; \nu+\frac{p+1}{2}; -RX)}{\Gamma_P(\alpha)|B|^{-\alpha} {}_3F_1(-\gamma, \nu+\delta+\gamma+\frac{p+1}{2}; \alpha; \nu+\frac{p+1}{2}; -RB^{-1})}$$

where $\mathrm{Re}(\alpha) > \frac{p-1}{2}$

=0 , elsewhere

gives a probability density function (p.d.f).

Special Cases: If we put R=I,the identity matrix of order p and B→0,and using result given by as Mathai and Saxena[1, ,5.2.35] ,we get

$$(2.3) \quad \int_{X>O} |X|^{\alpha-\frac{(p+1)}{2}} \frac{\Gamma_P(\gamma+\nu+\frac{(p+1)}{2})}{\Gamma_P(\nu+\frac{(p+1)}{2})} {}_2F_1(-\gamma, \nu+\delta+\gamma+\frac{p+1}{2}; \nu+\frac{p+1}{2}; -RX)$$

$$= \frac{\Gamma_P(\gamma+\nu+\frac{(p+1)}{2})\Gamma_P(\alpha)\Gamma_P(\nu+\delta+\gamma+\frac{p+1}{2}-\alpha)\Gamma_P(-\gamma-\alpha)}{\Gamma_P(\nu+\frac{(p+1)}{2}-\alpha)\Gamma_P(\nu+\delta+\gamma+\frac{p+1}{2})\Gamma_P(-\gamma)}$$

where $\mathrm{Re}(\alpha) > \frac{p+1}{2}, \mathrm{Re}(\gamma+\nu+\frac{(p+1)}{2}) > \frac{p-1}{2}$, $\mathrm{Re}(\alpha) > \frac{p-1}{2}, \mathrm{Re}(\nu+\frac{(p+1)}{2}) > \frac{p-1}{2}$

=0, elsewhere

Characteristic Function

The characteristic function $\phi_X(T)$ of X is given by

$$(2.4) \quad \phi_X(T) = E[\exp\{itr(XT)\}]$$
$$= \int_{X>0} \exp\{itr(XT)\} f(X) dX$$

where $i = (-1)^{\frac{1}{2}}$, T is $p \times p$ symmetric matrix of characteristic function variables and E denotes mathematical expectation.

by virtue of (2.4) ,we get characteristic function as

$$(2.5) \ \phi_X(T) = \frac{|B-iT|^{-\alpha} {}_3F_1[-\gamma, \nu+\delta+\gamma+\frac{p+1}{2}; \alpha; \nu+\frac{p+1}{2}; -R(B-iT)^{-1}]}{|B|^{-\alpha} {}_3F_1(-\gamma, \nu+\delta+\gamma+\frac{p+1}{2}; \alpha; \nu+\frac{p+1}{2}; -RB^{-1})}$$

where $\mathrm{Re}(\alpha) > \frac{p+1}{2}, \mathrm{Re}(\gamma+\nu+\frac{(p+1)}{2}) > \frac{p-1}{2}, \mathrm{Re}\, B > O, \mathrm{Re}(B) > O$

=0, elsewhere …

Moment Generating Function (m.g.f)

$$(2.6) \ E[etr(XT) = \int_{X>0} \exp\{tr(XT)\} f(X) dX$$

By virtue of (2.2), we get characteristic function as

$$E[etr(XT) = \frac{|B-T|^{-\alpha} {}_3F_1[-\gamma, \nu+\delta+\gamma+\frac{p+1}{2}; \alpha; \nu+\frac{p+1}{2}; -R(B-T)^{-1}]}{|B|^{-\alpha} {}_3F_1(-\gamma, \nu+\delta+\gamma+\frac{p+1}{2}; \alpha; \nu+\frac{p+1}{2}; -RB^{-1})}$$

where $(2.7) \quad \mathrm{Re}(\alpha) > \frac{p+1}{2}, \mathrm{Re}(\gamma+\nu+\frac{(p+1)}{2}) > \frac{p-1}{2}, \mathrm{Re}\, B > 0, \mathrm{Re}(B) > \mathrm{Re}(R)$

=0 , elsewhere

rthMoment of the Distribution

The r^{th} moment about origin of the matrix variable X with p.d.f (2.2)is given by

$$(2.8) E[|X|^r] = \int_{X>0} |X|^r f(X) dX$$

$$E[|X|^r] = \frac{|B|^{-r} {}_3F_1[-\gamma, \nu+\delta+\gamma+\frac{p+1}{2}; \alpha+r; \nu+\frac{p+1}{2}; -RB^{-1}]}{{}_3F_1(-\gamma, \nu+\delta+\gamma+\frac{p+1}{2}; \alpha; \nu+\frac{p+1}{2}; -RB^{-1})}$$

where $\mathrm{Re}(\alpha) > \frac{p+1}{2}, \mathrm{Re}(\gamma+\nu+\delta+\frac{(p+1)}{2}) > \frac{p-1}{2}, \mathrm{Re}\, B > O$

=0 , elsewhere

Mean and Variance of the distribution

$$E[|X|^1] = \int_{X>0} |X|^1 f(X) dX$$

$$(2.9)E[|X|^1]=\frac{|B|^{-1}{}_3F_1[-\gamma,\nu+\delta+\gamma+\frac{p+1}{2},\alpha+1;\nu+\frac{p+1}{2};-RB^{-1}]}{{}_3F_1(-\gamma,\nu+\delta+\gamma+\frac{p+1}{2},\alpha;\nu+\frac{p+1}{2};-RB^{-1})}$$

and

$$E[|X|^2]=\frac{|B|^{-1}{}_3F_1[-\gamma,\nu+\delta+\gamma+\frac{p+1}{2},\alpha+2;\nu+\frac{p+1}{2};-RB^{-1}]}{{}_3F_1(-\gamma,\nu+\delta+\gamma+\frac{p+1}{2},\alpha;\nu+\frac{p+1}{2};-RB^{-1})}$$

by definition, variance= $E(|X^2|)-[E(|X|)]^2$,we get

Variance=

(2.10)

$$\frac{|B|^{-2}\{[{}_3F_1(-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\alpha+2;\nu+\frac{p+1}{2};RB^{-1})-{}_3F_1[-\gamma,\nu+\delta+\gamma+\frac{p+1}{2},\alpha+1;\nu+\frac{p+1}{2};-RB^{-1}]\}}{[{}_3F_1(-\gamma,\nu+\delta+\gamma+\frac{p+1}{2},\alpha;\nu+\frac{p+1}{2};-RB^{-1})]^2}$$

$$\mathrm{Re}(\alpha)>\frac{p-1}{2}$$

Mellian and Laplace Transform of the Distribution

by definition of mellian transform is

$$(2.11)\quad M[f(X)]=\int_{X>0}|X|^{\delta-\frac{(p+1)}{2}}f(X)dX$$

$$=\frac{|B|^{-\delta+\frac{p+1}{2}}{}_3F_1[-\gamma,\nu+\delta+\gamma+\frac{p+1}{2},\alpha+\delta+\frac{p+1}{2};\nu+\frac{p+1}{2};-RB^{-1}]}{{}_3F_1(-\gamma,\nu+\delta+\gamma+\frac{p+1}{2},\alpha;\nu+\frac{p+1}{2};-RB^{-1})}$$

By definition of Laplace transform is

$$(2.12)\quad L[f(X)]=\int_{X>0}etr(-TX)f(X)dX$$

$$=\frac{|B+T|^{-\alpha}{}_3F_1[-\gamma,\nu+\delta+\gamma+\frac{p+1}{2},\alpha;\nu+\frac{p+1}{2};-R(B+T)^{-1}]}{|B|^{-\alpha}{}_3F_1(-\gamma,\nu+\delta+\gamma+\frac{p+1}{2},\alpha;\nu+\frac{p+1}{2};-RB^{-1})}$$

where $\mathrm{Re}(\alpha)>\frac{p+1}{2},\mathrm{Re}(\gamma+\nu+\frac{(p+1)}{2})>\frac{p-1}{2},\mathrm{Re}\,B>O,\mathrm{Re}(B)>O$

=0 , elsewhere .

Particular Cases: if we take Matrix of order 1×1 i.e if we take p=1 we and B=1,given matrix reduce to scalar variable i.e

$$\mathrm{f(x)} = \frac{|x|^{\alpha-\frac{(p+1)}{2}} etr(-bx)\,{}_2F_1(-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\nu+\frac{p+1}{2};-rx)}{\Gamma_P(\alpha)|b|^{-\alpha}\,{}_3F_1(-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\alpha;\nu+\frac{p+1}{2};-rb^{-1})}$$

$$,\mathrm{Re}(\alpha) > \frac{p+1}{2}, \mathrm{Re}(\gamma+\nu+\frac{(p+1)}{2}) > \frac{p-1}{2}$$

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